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Correspondence Analysis for First Degree Entailment

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In this paper natural deduction systems for four-valued logic **FDE** (first degree entailment) and its extensions are constructed. At that B. Kooi and A. Tamminga's method of correspondence analysis is used. All possible four-valued unary ($\star$) and binary ($\circ$) propositional connectives which could be added to **FDE** are considered. Then **FDE** is extended by Boolean negation ($\sim$) and every entry (line) of truth tables for $\star$ and $\circ$ is characterized by inference scheme. By adding all inference schemes characterizing truth tables for $\star$ and $\circ$ as rules of inference to the natural deduction for **FDE**, natural deduction for extension of **FDE** is obtained. In addition, applying of correspondence analysis gives axiomatizations of implicative extensions of **FDE** including $\mathbf{BN_4}$ and some extensions by classical implications.

Keywords: correspondence analysis, natural deduction, first degree entailment, Belnap-Dunn logic, four-valued logic, implicative extensions, classical implication

1. Introduction

A history of the logic **FDE** dates back to N.D. Belnap's abstract [5] and A.R. Anderson and N.D. Belnap's paper [1]. They investigate a system of first degree (tautological) entailment which inferences avoid paradoxes of classical entailment and contain connectives $\neg$, $\wedge$, and $\vee$. An implication is occured in a formula only once: as the main connective. In other words, all first degree formulas are of the form $A \to B$, where A and B do not contain $\to$. Since the definitions of $\to$ and $\models$ are equivalent, $\to$ is replaced by $\models$ in many papers on this subject (including this one). Moreover, Anderson and Belnap proved that **FDE** is a first degree fragment of relevant logic **E**, i.e. $A \to B$ (where A and B don't contain $\to$) is provable in **E** iff $A \to B$ is a first degree (tautological) entailment.

There are various semantics for **FDE**, but in this paper only two of them will be need: N.D. Belnap's semantics [3, 4] and J.M. Dunn's one [8]. They will be discussed in the next section.

The first formalisation of **FDE** was introduced in [1]. Since then, various studies of proof systems for **FDE** have been carried out. For this paper G. Priest's monography [15] is of particular importance: natural deduction system built in it is actively used here. It should be noted that there are also investigations devoted to extensions of **FDE**. (Some of them are considered in the section 6.)

This paper is a kind of continuation and generalization of these studies: it is an attempt to explore natural deductions systems axiomatizing all possible truth-table expansions of **FDE**[1]. Solving this problem, I use the technique of correspondence analysis, first applied by B. Kooi and A. Tamminga [13] for three-valued logic $\mathbf{K_3^2}$ (**LP**) [16] and its extensions[2].

In [20] A. Tamminga explains the idea of correspondence analysis applied to **LP** as follows:

> <...> characterize every possible single entry in the truth table of a unary or a binary truth-functional operator by a basic inference scheme. As a consequence, each unary and each binary truth-functional operator is characterized by a set of basic inference schemes. Kooi and Tamminga show that if we add the inference schemes that characterize an operator to a natural deduction system for **LP**, we immediately obtain a natural deduction system that is sound and complete with respect to the logic that contains, next to **LP**'s negation, disjunction, and conjunction, the additional operator [20, p. 256].

Thus, this paper continues B. Kooi and A. Tamminga's proof-theoretic studies of three-valued logics, spreading them on the field of four-valued logics and thereby offering universal instrument of axiomatization of all possible truth-table extensions of **FDE+**.

2. Semantics

N.D. Belnap's semantics [3, 4]. Consider a matrix $\mathfrak{M}_4 = \langle\{1, b, n, 0\}, \neg, \wedge, \vee, \{1, b\}\rangle$ of the logic **FDE**, a matrix $\mathfrak{M}_4^+ = \langle\{1, b, n, 0\}, \neg, \sim, \wedge, \vee, \{1, b\rangle\}$ of the logic **FDE+**, and a matrix $\mathfrak{M}_4^\# = \langle\{1, b, n, 0\}, \neg, \sim, \wedge, \vee, \star_1, \dots, \star_n, \circ_1, \dots, \circ_m, \{1, b\rangle\}$ of the logic **FDE#**.

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[1]Note that for some technical reasons before constructing such systems **FDE** (alphabet of which language contains $\neg$ (De Morgan negation), $\wedge$ (conjunction) and $\vee$ (disjunction)) should be expanded by Boolean negation $\sim$. Let us denote this logic through **FDE+**.

[2]In A. Tamminga's paper [20] the similar result is obtained for $\mathbf{K_3}$ [12, 11] and its extensions.

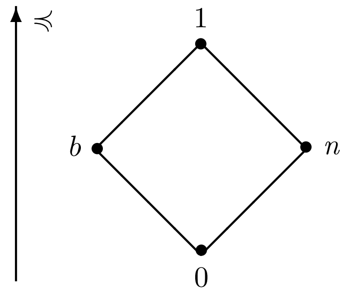
A	$\neg$	$\sim$
1	0	0
b	b	n
n	n	b
0	1	1

$\wedge$	1	b	n	0
1	1	b	n	0
b	b	b	0	0
n	n	0	n	0
0	0	0	0	0

$\vee$	1	b	n	0
1	1	1	1	1
b	1	b	1	b
n	1	1	n	n
0	1	b	n	0

Unary operators $\star_1, \ldots, \star_n$ and binary operators $\circ_1, \ldots, \circ_m$ are *arbitrary*. In the particular case they can be connectives of **FDE**+ or all possible four-valued connectives. By these reason I do not give here truth tables for them.

The values are ordered as follows: $0 \preccurlyeq n$, $0 \preccurlyeq b$, $n \preccurlyeq 1$, $b \preccurlyeq 1$; n and b are incomparable (see a picture below).



It is natural to regard the value 1 as "true", b as "true and false", n as "not true and not false" and 0 as "false". Note that N.D. Belnap himself defined the entailment through the relation $\preccurlyeq$. It was J.M. Font [9] who first proved that it is possible to redefine the entailment relation through designated values. The same result was independently obtained by D.V. Zaitsev and Y.V. Shramko [22]. Moreover, Y. Shramko and H. Wansing [18] proved that it is possible to define entailment through set $\{0, b\}$ of *anti*designated values.

J.M. Dunn's semantics [8]. Truth values here are subsets of a set of classical truth values $\{\mathfrak{t}, \mathfrak{f}\}$, that is $\{\mathfrak{t}\}, \{\mathfrak{t}, \mathfrak{f}\}, \emptyset$ and $\{\mathfrak{f}\}$ which are analogues of values 1, b, n and 0 from Belnap's semantics. The conditions of truth and falsity for formulas are as follows (v is a valuation):

$$\mathfrak{t} \in v(\neg A) \Leftrightarrow \mathfrak{f} \in v(A);$$
$$\mathfrak{f} \in v(\neg A) \Leftrightarrow \mathfrak{t} \in v(A);$$
$$\mathfrak{t} \in v(\sim A) \Leftrightarrow \mathfrak{t} \notin v(A);$$
$$\mathfrak{f} \in v(\sim A) \Leftrightarrow \mathfrak{f} \notin v(A);$$
$$\mathfrak{t} \in v(A \wedge B) \Leftrightarrow \mathfrak{t} \in v(A) \mathbin{\dot{\wedge}} \mathfrak{t} \in v(B);$$
$$\mathfrak{f} \in v(A \wedge B) \Leftrightarrow \mathfrak{f} \in v(A) \mathbin{\dot{\vee}} \mathfrak{f} \in v(B);$$
$$\mathfrak{t} \in v(A \vee B) \Leftrightarrow \mathfrak{t} \in v(A) \mathbin{\dot{\vee}} \mathfrak{t} \in v(B);$$
$$\mathfrak{f} \in v(A \vee B) \Leftrightarrow \mathfrak{f} \in v(A) \mathbin{\dot{\wedge}} \mathfrak{f} \in v(B).$$

In terms of J.M. Dunn's semantics the relation of entailment in logics **FDE**, **FDE+** and **FDE#** is defined as follows:

$$\Gamma \models A \Leftrightarrow \dot{\forall} v(\dot{\forall}_{B \in \Gamma} B\ \mathsf{t} \in v(B) \Rightarrow \mathsf{t} \in v(A)).$$

3. Inference schemes for arbitrary connectives

Remark 1 (About designations). Let us denote through $\mathcal{L}^{\#}$ a language of **FDE#**, through *Prop* a set of all propositional variables of the language $\mathcal{L}^{\#}$, through $Form^{\#}$ a set of all $\mathcal{L}^{\#}$-formulas (formulas in the language $\mathcal{L}^{\#}$); through $f_\star$ a truth table for $\star$ and through $f_\circ$ a truth table for $\circ$. Let $x, y, z \in \{1, b, n, 0\}$, then let us denote through $f_\star(x) = y$ such entry (line) of a truth table $f_\star$ that $\dot{\forall} A \dot{\forall} v(v(A) = x \Rightarrow v(\star A) = y)$; and through $f_\circ(x, y) = z$ such entry of a truth table $f_\circ$ that $\dot{\forall} A \dot{\forall} v((v(A) = x \mathbin{\dot{\wedge}} v(B) = y) \Rightarrow v(A \circ B) = z)$.

So, in this section propositions 1 and 2 are formulated. The first one states that for every entry of the form $f_\star(x) = y$ a characteristic inference scheme corresponds. The second one states that for every entry of the form $f_\circ(x, y) = z$ a characteristic inference scheme corresponds. It is clear that every operator $\star$ has 4 entries and it is characterised by 4 inference schemes; and every operator $\circ$ has 16 entries and it is characterised by 16 inference schemes. In the section 5 it is proved that by adding all inference schemes which characterise operators $\star_1, \ldots, \star_n, \circ_1, \ldots, \circ_m$ as rules of inference to **FDE+** we get not only sound, but complete natural deduction system for **FDE#** (i.e. for **FDE+** extended by $\star_1, \ldots, \star_n, \circ_1, \ldots, \circ_m$).

Proposition 1. *For every $\mathcal{L}^{\#}$-formula A:*

$$f_\star(0) = \begin{cases} 0 & \Leftrightarrow \quad {\sim} A, \neg A \models {\sim}\star A \wedge \neg \star A \\ n & \Leftrightarrow \quad {\sim} A, \neg A \models {\sim}\star A \wedge {\sim}\neg \star A \\ b & \Leftrightarrow \quad {\sim} A, \neg A \models \star A \wedge \neg \star A \\ 1 & \Leftrightarrow \quad {\sim} A, \neg A \models \star A \wedge {\sim}\neg \star A \end{cases}$$

$$f_\star(n) = \begin{cases} 0 & \Leftrightarrow \quad {\sim} A, {\sim}\neg A \models {\sim}\star A \wedge \neg \star A \\ n & \Leftrightarrow \quad {\sim} A, {\sim}\neg A \models {\sim}\star A \wedge {\sim}\neg \star A \\ b & \Leftrightarrow \quad {\sim} A, {\sim}\neg A \models \star A \wedge \neg \star A \\ 1 & \Leftrightarrow \quad {\sim} A, {\sim}\neg A \models \star A \wedge {\sim}\neg \star A \end{cases}$$

$$f_\star(b) = \begin{cases} 0 & \Leftrightarrow \quad A, \neg A \models {\sim}\star A \wedge \neg \star A \\ n & \Leftrightarrow \quad A, \neg A \models {\sim}\star A \wedge {\sim}\neg \star A \\ b & \Leftrightarrow \quad A, \neg A \models \star A \wedge \neg \star A \\ 1 & \Leftrightarrow \quad A, \neg A \models \star A \wedge {\sim}\neg \star A \end{cases}$$

$$f_\star(1) = \begin{cases} 0 & \Leftrightarrow \quad A, \sim\neg A \models \sim\star A \wedge \neg\star A \\ n & \Leftrightarrow \quad A, \sim\neg A \models \sim\star A \wedge \sim\neg\star A \\ b & \Leftrightarrow \quad A, \sim\neg A \models \star A \wedge \neg\star A \\ 1 & \Leftrightarrow \quad A, \sim\neg A \models \star A \wedge \sim\neg\star A \end{cases}$$

Proof. Suppose $f_\star(0) = 1$. Let us show that $\dot\forall A$: $\sim A, \neg A \models \star A \wedge \sim\neg\star A$. According to the remark 1, $f_\star(0) = 1$ means that $f_\star$ has an entry such that $\dot\forall A \dot\forall v(v(A) = 0 \Rightarrow v(\star A) = 1)$. In the terms of J.M. Dunn's semantics the last statement is interpreted as (α) $\dot\forall A \dot\forall v((\mathfrak{t} \notin v(A) \mathbin{\dot\wedge} \mathfrak{f} \in v(A)) \Rightarrow (\mathfrak{t} \in v(\star A) \mathbin{\dot\wedge} \mathfrak{f} \notin v(\star A)))$. Now suppose (β) $\mathfrak{t} \in v(\sim A)$ and $\mathfrak{t} \in v(\neg A)$. Therefore, (γ) $\mathfrak{t} \notin v(A)$ and $\mathfrak{f} \in v(A)$. From (α) and (γ) obtain that (δ) $\mathfrak{t} \in v(\star A) \mathbin{\dot\wedge} \mathfrak{f} \notin v(\star A))$. Hence, (ε) $\mathfrak{t} \in v(\star A \wedge \sim\neg\star A)$. From (β) and (ε) obtain (ζ) $\dot\forall A \dot\forall v((\mathfrak{t} \in v(\sim A) \mathbin{\dot\wedge} \mathfrak{t} \in v(\neg A)) \Rightarrow \mathfrak{t} \in v(\star A \wedge \sim\neg\star A))$. Therefore, (η) $\dot\forall A$: $\sim A, \neg A \models \star A \wedge \sim\neg\star A$.

Suppose (θ) $\dot\forall A$: $\sim A, \neg A \models \star A \wedge \sim\neg\star A$, let us prove that $f_\star(0) = 1$. From (θ) obtain (ι) $\dot\forall A \dot\forall v((\mathfrak{t} \in v(\sim A) \mathbin{\dot\wedge} \mathfrak{t} \in v(\neg A)) \Rightarrow \mathfrak{t} \in v(\star A \wedge \sim\neg\star A))$. From (ι) follows (κ) $\dot\forall A \dot\forall v((\mathfrak{t} \notin v(A) \mathbin{\dot\wedge} \mathfrak{f} \in v(A)) \Rightarrow (\mathfrak{t} \in v(\star A) \mathbin{\dot\wedge} \mathfrak{f} \notin v(\star A)))$, which is equivalent to (λ) $\dot\forall A \dot\forall v(v(A) = 0 \Rightarrow v(\star A) = 1)$. According to the remark 1, $f_\star(0) = 1$ is an abbreviation for (λ).

The other cases are proved similarly. □

Now let us formulate the analogues proposition for binary operators.

Proposition 2. *For every $\mathcal{L}^{\#}$-formulas A and B:*

$$f_\circ(0,0) = \begin{cases} 0 & \Leftrightarrow \quad \sim A, \neg A, \sim B, \neg B \models \sim(A \circ B) \wedge \neg(A \circ B) \\ n & \Leftrightarrow \quad \sim A, \neg A, \sim B, \neg B \models \sim(A \circ B) \wedge \sim\neg(A \circ B) \\ b & \Leftrightarrow \quad \sim A, \neg A, \sim B, \neg B \models (A \circ B) \wedge \neg(A \circ B) \\ 1 & \Leftrightarrow \quad \sim A, \neg A, \sim B, \neg B \models (A \circ B) \wedge \sim\neg(A \circ B) \end{cases}$$

$$f_\circ(0,n) = \begin{cases} 0 & \Leftrightarrow \quad \sim A, \neg A, \sim B, \sim\neg B \models \sim(A \circ B) \wedge \neg(A \circ B) \\ n & \Leftrightarrow \quad \sim A, \neg A, \sim B, \sim\neg B \models \sim(A \circ B) \wedge \sim\neg(A \circ B) \\ b & \Leftrightarrow \quad \sim A, \neg A, \sim B, \sim\neg B \models (A \circ B) \wedge \neg(A \circ B) \\ 1 & \Leftrightarrow \quad \sim A, \neg A, \sim B, \sim\neg B \models (A \circ B) \wedge \sim\neg(A \circ B) \end{cases}$$

$$f_\circ(0,b) = \begin{cases} 0 & \Leftrightarrow \quad \sim A, \neg A, B, \neg B \models \sim(A \circ B) \wedge \neg(A \circ B) \\ n & \Leftrightarrow \quad \sim A, \neg A, B, \neg B \models \sim(A \circ B) \wedge \sim\neg(A \circ B) \\ b & \Leftrightarrow \quad \sim A, \neg A, B, \neg B \models (A \circ B) \wedge \neg(A \circ B) \\ 1 & \Leftrightarrow \quad \sim A, \neg A, B, \neg B \models (A \circ B) \wedge \sim\neg(A \circ B) \end{cases}$$

$$f_\circ(0,1) = \begin{cases} 0 & \Leftrightarrow \quad \sim A, \neg A, B, \sim\neg B \models \sim(A \circ B) \wedge \neg(A \circ B) \\ n & \Leftrightarrow \quad \sim A, \neg A, B, \sim\neg B \models \sim(A \circ B) \wedge \sim\neg(A \circ B) \\ b & \Leftrightarrow \quad \sim A, \neg A, B, \sim\neg B \models (A \circ B) \wedge \neg(A \circ B) \\ 1 & \Leftrightarrow \quad \sim A, \neg A, B, \sim\neg B \models (A \circ B) \wedge \sim\neg(A \circ B) \end{cases}$$

$$f_\circ(n,0) = \begin{cases} 0 & \Leftrightarrow & {\sim}A,\, {\sim}\neg A,\, {\sim}B,\, \neg B \models {\sim}(A \circ B) \wedge \neg(A \circ B) \\ n & \Leftrightarrow & {\sim}A,\, {\sim}\neg A,\, {\sim}B,\, \neg B \models {\sim}(A \circ B) \wedge {\sim}\neg(A \circ B) \\ b & \Leftrightarrow & {\sim}A,\, {\sim}\neg A,\, {\sim}B,\, \neg B \models (A \circ B) \wedge \neg(A \circ B) \\ 1 & \Leftrightarrow & {\sim}A,\, {\sim}\neg A,\, {\sim}B,\, \neg B \models (A \circ B) \wedge {\sim}\neg(A \circ B) \end{cases}$$

$$f_\circ(n,n) = \begin{cases} 0 & \Leftrightarrow & {\sim}A,\, {\sim}\neg A,\, {\sim}B,\, {\sim}\neg B \models {\sim}(A \circ B) \wedge \neg(A \circ B) \\ n & \Leftrightarrow & {\sim}A,\, {\sim}\neg A,\, {\sim}B,\, {\sim}\neg B \models {\sim}(A \circ B) \wedge {\sim}\neg(A \circ B) \\ b & \Leftrightarrow & {\sim}A,\, {\sim}\neg A,\, {\sim}B,\, {\sim}\neg B \models (A \circ B) \wedge \neg(A \circ B) \\ 1 & \Leftrightarrow & {\sim}A,\, {\sim}\neg A,\, {\sim}B,\, {\sim}\neg B \models (A \circ B) \wedge {\sim}\neg(A \circ B) \end{cases}$$

$$f_\circ(n,b) = \begin{cases} 0 & \Leftrightarrow & {\sim}A,\, {\sim}\neg A,\, B,\, \neg B \models {\sim}(A \circ B) \wedge \neg(A \circ B) \\ n & \Leftrightarrow & {\sim}A,\, {\sim}\neg A,\, B,\, \neg B \models {\sim}(A \circ B) \wedge {\sim}\neg(A \circ B) \\ b & \Leftrightarrow & {\sim}A,\, {\sim}\neg A,\, B,\, \neg B \models (A \circ B) \wedge \neg(A \circ B) \\ 1 & \Leftrightarrow & {\sim}A,\, {\sim}\neg A,\, B,\, \neg B \models (A \circ B) \wedge {\sim}\neg(A \circ B) \end{cases}$$

$$f_\circ(n,1) = \begin{cases} 0 & \Leftrightarrow & {\sim}A,\, {\sim}\neg A,\, B,\, {\sim}\neg B \models {\sim}(A \circ B) \wedge \neg(A \circ B) \\ n & \Leftrightarrow & {\sim}A,\, {\sim}\neg A,\, B,\, {\sim}\neg B \models {\sim}(A \circ B) \wedge {\sim}\neg(A \circ B) \\ b & \Leftrightarrow & {\sim}A,\, {\sim}\neg A,\, B,\, {\sim}\neg B \models (A \circ B) \wedge \neg(A \circ B) \\ 1 & \Leftrightarrow & {\sim}A,\, {\sim}\neg A,\, B,\, {\sim}\neg B \models (A \circ B) \wedge {\sim}\neg(A \circ B) \end{cases}$$

$$f_\circ(b,0) = \begin{cases} 0 & \Leftrightarrow & A,\, \neg A,\, {\sim}B,\, \neg B \models {\sim}(A \circ B) \wedge \neg(A \circ B) \\ n & \Leftrightarrow & A,\, \neg A,\, {\sim}B,\, \neg B \models {\sim}(A \circ B) \wedge {\sim}\neg(A \circ B) \\ b & \Leftrightarrow & A,\, \neg A,\, {\sim}B,\, \neg B \models (A \circ B) \wedge \neg(A \circ B) \\ 1 & \Leftrightarrow & A,\, \neg A,\, {\sim}B,\, \neg B \models (A \circ B) \wedge {\sim}\neg(A \circ B) \end{cases}$$

$$f_\circ(b,n) = \begin{cases} 0 & \Leftrightarrow & A,\, \neg A,\, {\sim}B,\, {\sim}\neg B \models {\sim}(A \circ B) \wedge \neg(A \circ B) \\ n & \Leftrightarrow & A,\, \neg A,\, {\sim}B,\, {\sim}\neg B \models {\sim}(A \circ B) \wedge {\sim}\neg(A \circ B) \\ b & \Leftrightarrow & A,\, \neg A,\, {\sim}B,\, {\sim}\neg B \models (A \circ B) \wedge \neg(A \circ B) \\ 1 & \Leftrightarrow & A,\, \neg A,\, {\sim}B,\, {\sim}\neg B \models (A \circ B) \wedge {\sim}\neg(A \circ B) \end{cases}$$

$$f_\circ(b,b) = \begin{cases} 0 & \Leftrightarrow & A,\, \neg A,\, B,\, \neg B \models {\sim}(A \circ B) \wedge \neg(A \circ B) \\ n & \Leftrightarrow & A,\, \neg A,\, B,\, \neg B \models {\sim}(A \circ B) \wedge {\sim}\neg(A \circ B) \\ b & \Leftrightarrow & A,\, \neg A,\, B,\, \neg B \models (A \circ B) \wedge \neg(A \circ B) \\ 1 & \Leftrightarrow & A,\, \neg A,\, B,\, \neg B \models (A \circ B) \wedge {\sim}\neg(A \circ B) \end{cases}$$

$$f_\circ(b,1) = \begin{cases} 0 & \Leftrightarrow & A,\, \neg A,\, B,\, {\sim}\neg B \models {\sim}(A \circ B) \wedge \neg(A \circ B) \\ n & \Leftrightarrow & A,\, \neg A,\, B,\, {\sim}\neg B \models {\sim}(A \circ B) \wedge {\sim}\neg(A \circ B) \\ b & \Leftrightarrow & A,\, \neg A,\, B,\, {\sim}\neg B \models (A \circ B) \wedge \neg(A \circ B) \\ 1 & \Leftrightarrow & A,\, \neg A,\, B,\, {\sim}\neg B \models (A \circ B) \wedge {\sim}\neg(A \circ B) \end{cases}$$

$$f_\circ(1,0) = \begin{cases} 0 & \Leftrightarrow & A,\, {\sim}\neg A,\, {\sim}B,\, \neg B \models {\sim}(A \circ B) \wedge \neg(A \circ B) \\ n & \Leftrightarrow & A,\, {\sim}\neg A,\, {\sim}B,\, \neg B \models {\sim}(A \circ B) \wedge {\sim}\neg(A \circ B) \\ b & \Leftrightarrow & A,\, {\sim}\neg A,\, {\sim}B,\, \neg B \models (A \circ B) \wedge \neg(A \circ B) \\ 1 & \Leftrightarrow & A,\, {\sim}\neg A,\, {\sim}B,\, \neg B \models (A \circ B) \wedge {\sim}\neg(A \circ B) \end{cases}$$

$$f_\circ(1,n) = \begin{cases} 0 & \Leftrightarrow \quad A,\ {\sim}\neg A,\ {\sim}B,\ {\sim}\neg B \models {\sim}(A\circ B) \wedge \neg(A\circ B) \\ n & \Leftrightarrow \quad A,\ {\sim}\neg A,\ {\sim}B,\ {\sim}\neg B \models {\sim}(A\circ B) \wedge {\sim}\neg(A\circ B) \\ b & \Leftrightarrow \quad A,\ {\sim}\neg A,\ {\sim}B,\ {\sim}\neg B \models (A\circ B) \wedge \neg(A\circ B) \\ 1 & \Leftrightarrow \quad A,\ {\sim}\neg A,\ {\sim}B,\ {\sim}\neg B \models (A\circ B) \wedge {\sim}\neg(A\circ B) \end{cases}$$

$$f_\circ(1,b) = \begin{cases} 0 & \Leftrightarrow \quad A,\ {\sim}\neg A,\ B,\ \neg B \models {\sim}(A\circ B) \wedge \neg(A\circ B) \\ n & \Leftrightarrow \quad A,\ {\sim}\neg A,\ B,\ \neg B \models {\sim}(A\circ B) \wedge {\sim}\neg(A\circ B) \\ b & \Leftrightarrow \quad A,\ {\sim}\neg A,\ B,\ \neg B \models (A\circ B) \wedge \neg(A\circ B) \\ 1 & \Leftrightarrow \quad A,\ {\sim}\neg A,\ B,\ \neg B \models (A\circ B) \wedge {\sim}\neg(A\circ B) \end{cases}$$

$$f_\circ(1,1) = \begin{cases} 0 & \Leftrightarrow \quad A,\ {\sim}\neg A,\ B,\ {\sim}\neg B \models {\sim}(A\circ B) \wedge \neg(A\circ B) \\ n & \Leftrightarrow \quad A,\ {\sim}\neg A,\ B,\ {\sim}\neg B \models {\sim}(A\circ B) \wedge {\sim}\neg(A\circ B) \\ b & \Leftrightarrow \quad A,\ {\sim}\neg A,\ B,\ {\sim}\neg B \models (A\circ B) \wedge \neg(A\circ B) \\ 1 & \Leftrightarrow \quad A,\ {\sim}\neg A,\ B,\ {\sim}\neg B \models (A\circ B) \wedge {\sim}\neg(A\circ B) \end{cases}$$

PROOF. Suppose $f_\circ(b,n) = 0$. Let us show that $\dot\forall A \dot\forall B$: A, $\neg A$, ${\sim}B$, ${\sim}\neg B \models {\sim}(A\circ B) \wedge \neg(A\circ B)$. According to the remark 1, $f_\circ(b,n) = 0$ means that $f_\circ$ has an entry such that $\dot\forall A \dot\forall B \dot\forall v((v(A) = b \mathbin{\dot\wedge} v(B) = n) \Rightarrow v(A\circ B) = 0)$. In the terms of J.M. Dunn's semantics the last statement is understood as (α) $\dot\forall A \dot\forall B \dot\forall v((\mathfrak{t} \in v(A) \mathbin{\dot\wedge} \mathfrak{f} \in v(A) \mathbin{\dot\wedge} \mathfrak{t} \notin v(B) \mathbin{\dot\wedge} \mathfrak{f} \notin v(B)) \Rightarrow (\mathfrak{t} \notin v(A\circ B) \mathbin{\dot\wedge} \mathfrak{f} \in v(A\circ B)))$. Now suppose (β) $\mathfrak{t} \in v(A)$, $\mathfrak{t} \in v(\neg A)$, $\mathfrak{t} \in v({\sim}B)$ and $\mathfrak{t} \in v({\sim}\neg B)$. Therefore, (γ) $\mathfrak{t} \in v(A) \mathbin{\dot\wedge} \mathfrak{f} \in v(A) \mathbin{\dot\wedge} \mathfrak{t} \notin v(B) \mathbin{\dot\wedge} \mathfrak{f} \notin v(B)$. From (α) and (γ) obtain that (δ) $\mathfrak{t} \notin v(A\circ B) \mathbin{\dot\wedge} \mathfrak{f} \in v(A\circ B)$. Hence, (ε) $\mathfrak{t} \in v({\sim}(A\circ B) \wedge \neg(A\circ B))$. From (β) and (ε) obtain (ζ) $\dot\forall A \dot\forall v((\mathfrak{t} \in v(A) \mathbin{\dot\wedge} \mathfrak{t} \in v(\neg A) \mathbin{\dot\wedge} \mathfrak{t} \in v({\sim}B) \mathbin{\dot\wedge} \mathfrak{t} \in v({\sim}\neg B)) \Rightarrow \mathfrak{t} \in v({\sim}(A\circ B) \wedge \neg(A\circ B)))$. Therefore, (η) $\dot\forall A \dot\forall B$: A, $\neg A$, ${\sim}B$, ${\sim}\neg B \models {\sim}(A\circ B) \wedge \neg(A\circ B)$.

Suppose (θ) $\dot\forall A \dot\forall B$: A, $\neg A$, ${\sim}B$, ${\sim}\neg B \models {\sim}(A\circ B) \wedge \neg(A\circ B)$, let us prove that $f_\circ(b,n) = 0$. From (θ) obtain (ι) $\dot\forall A \dot\forall v((\mathfrak{t} \in v(A) \mathbin{\dot\wedge} \mathfrak{t} \in v(\neg A) \mathbin{\dot\wedge} \mathfrak{t} \in v({\sim}B) \mathbin{\dot\wedge} \mathfrak{t} \in v({\sim}\neg B)) \Rightarrow \mathfrak{t} \in v({\sim}(A\circ B) \wedge \neg(A\circ B)))$. From (ι) follows (κ) $\dot\forall A \dot\forall B \dot\forall v((\mathfrak{t} \in v(A) \mathbin{\dot\wedge} \mathfrak{f} \in v(A) \mathbin{\dot\wedge} \mathfrak{t} \notin v(B) \mathbin{\dot\wedge} \mathfrak{f} \notin v(B)) \Rightarrow (\mathfrak{t} \notin v(A\circ B) \mathbin{\dot\wedge} \mathfrak{f} \in v(A\circ B)))$, which is equivalent to (λ) $\dot\forall A \dot\forall B \dot\forall v((v(A) = b \mathbin{\dot\wedge} v(B) = n) \Rightarrow v(A\circ B) = 0)$. According to the remark 1, $f_\circ(b,n) = 0$ is an abbreviation for (λ).

The other cases are proved similarly. □

4. Natural deduction system

A natural deduction system for **FDE** is as follows[3]:

$$(\neg\neg I)\ \frac{A}{\neg\neg A} \qquad (\neg\neg E)\ \frac{\neg\neg A}{A} \qquad (\vee I_1)\ \frac{A}{A\vee B} \qquad (\vee I_2)\ \frac{B}{A\vee B}$$

[3]This system was first introduced by G. Priest [15].

$$(\vee E)\ \frac{A\vee B,\quad \begin{matrix}[A]^{+1}\\ C,\end{matrix}\quad \begin{matrix}[B]^{+2}\\ C\end{matrix}}{C} \qquad (\wedge I)\ \frac{A,\ B}{A\wedge B}$$

$$(\wedge E_1)\ \frac{A\wedge B}{A} \qquad (\wedge E_2)\ \frac{A\wedge B}{B} \qquad (\neg\vee I)\ \frac{\neg A\wedge\neg B}{\neg(A\vee B)}$$

$$(\neg\vee E)\ \frac{\neg(A\vee B)}{\neg A\wedge\neg B} \qquad (\neg\wedge I)\ \frac{\neg A\vee\neg B}{\neg(A\wedge B)} \qquad (\neg\wedge E)\ \frac{\neg(A\wedge B)}{\neg A\vee\neg B}$$

Rules for Boolean negation are as follows:

$$(EFQ)\ \frac{A,\ \sim A}{B} \qquad (EM)\ \frac{}{A\vee\sim A} \qquad (\sim\neg E)\ \frac{\sim\neg A}{\neg\sim A} \qquad (\neg\sim E)\ \frac{\neg\sim A}{\sim\neg A}$$

A rule of inference of the form $\mathcal{R}_\star(x,y)\ \dfrac{A_1,\dots,A_n}{B}$ corresponds to an entry $f_\star(x)=y$ of a truth table $f_\star$ and a rule of the form $\mathcal{R}_\circ(x,y,z)$ $\dfrac{A_1,\dots,A_m}{B}$ corresponds to an entry $f_\circ(x,y)=z$ of a truth table $f_\circ$. Each connective $\star$ needs 4 rules of the form $\mathcal{R}_\star(x,y)$ and each connective $\circ$ needs 16 rules of the form $\mathcal{R}_\circ(x,y,z)$. These rules are inference schemes introduced in the section 3. Here is an example. According to the proposition 1, the rule $\mathcal{R}_\star(0,0)$ corresponds to the entry $f_\star(0)=0$ of the truth table $f_\star$:

$$\mathcal{R}_\star(0,0)\ \frac{\sim A,\ \neg A}{\sim\star A\wedge\neg\star A}\ .$$

5. Completeness theorem

It is not difficult to prove the following theorem 1.

THEOREM 1 (SOUNDNESS). *For every set of $\mathcal{L}^{\#}$-formulas Γ and for every $\mathcal{L}^{\#}$-formula A: $\Gamma\vdash A\Rightarrow\Gamma\models A$.*

While proving completeness theorem prime theories are used as syntactic analogues of valuations.

DEFINITION 1. For every set of $\mathcal{L}^{\#}$-formulas Γ and for every $\mathcal{L}^{\#}$-formulas A and B Γ is a prime theory, if the following conditions are true:

(Γ1) $\Gamma\neq Form^{\#}$ (*non-triviality*);

(Γ2) $\Gamma\vdash A\Leftrightarrow A\in\Gamma$ (*closure of* $\vdash$);

(Γ3) $A\vee B\in\Gamma\Rightarrow(A\in\Gamma\ \dot\vee\ B\in\Gamma)$ (*primeness*).

Elementhoods of $\mathcal{L}^{\#}$-formulas in prime theories are used as syntactic analogues of truth values.

Definition 2. For every prime theory Γ and for every $\mathcal{L}^{\#}$-formula A let us call $e(A, \Gamma)$ an elementhood of A in Γ and define it as follows:

$$e(A,\Gamma) = \begin{cases} 1 & \Leftrightarrow \quad A \in \Gamma,\ \neg A \notin \Gamma; \\ b & \Leftrightarrow \quad A \in \Gamma,\ \neg A \in \Gamma; \\ n & \Leftrightarrow \quad A \notin \Gamma,\ \neg A \notin \Gamma; \\ 0 & \Leftrightarrow \quad A \notin \Gamma,\ \neg A \in \Gamma. \end{cases}$$

The following lemma 1 shows us that the definition 2 is consistent with the truth tables for the propositional connectives.

Lemma 1. For every prime theory Γ and for every $\mathcal{L}^{\#}$-formulas A and B:

(1) $f_{\neg}(e(A,\Gamma)) = e(\neg A,\Gamma)$;

(2) $f_{\sim}(e(A,\Gamma)) = e(\sim A,\Gamma)$;

(3) $f_{\vee}(e(A,\Gamma), e(B,\Gamma)) = e(A \vee B,\Gamma)$;

(4) $f_{\wedge}(e(A,\Gamma), e(B,\Gamma)) = e(A \wedge B,\Gamma)$;

(5) $f_{\star}(e(A,\Gamma)) = e(\star A,\Gamma)$;

(6) $f_{\circ}(e(A,\Gamma), e(B,\Gamma)) = e(A \circ B,\Gamma)$.

Proof.

(1) (A) $e(A,\Gamma) = 0$. Then $A \notin \Gamma$, $\neg A \in \Gamma$. Suppose $\neg\neg A \in \Gamma$. According to $(\neg\neg E)$, $A \in \Gamma$. Contradiction. Hence, $\neg\neg A \notin \Gamma$. Therefore, $e(\neg A,\Gamma) = 1 = f_{\neg}(0) = f_{\neg}(e(A,\Gamma))$.

(B) $e(A,\Gamma) = n$. Then $A \notin \Gamma$, $\neg A \notin \Gamma$. Similar to (A).

(C) $e(A,\Gamma) = b$. Then $A \in \Gamma$, $\neg A \in \Gamma$. According to $(\neg\neg I)$, $\neg\neg A \in \Gamma$. Therefore, $e(\neg A,\Gamma) = n = f_{\neg}(n) = f_{\neg}(e(A,\Gamma))$.

(D) $e(A,\Gamma) = 1$. Then $A \in \Gamma$, $\neg A \notin \Gamma$. Similar to (C).

(2) (A) $e(A,\Gamma) = 0$. $A \notin \Gamma$, $\neg A \in \Gamma$. According to (EM) and $(\Gamma 3)$, $A \in \Gamma \ \dot{\vee} \sim A \in \Gamma$. Since $A \notin \Gamma$, $\sim A \in \Gamma$. Let $\neg \sim A \in \Gamma$. By the rule $(\neg \sim E)$ $\sim\neg A \in \Gamma$, by the rule (EFQ) $B \in \Gamma$. Contradiction. $\neg \sim A \notin \Gamma$. Hence, $e(\sim A,\Gamma) = 1 = f_{\sim}(0) = f_{\sim}(e(A,\Gamma))$.

(B) $e(A,\Gamma) = n$. $A \notin \Gamma$, $\neg A \notin \Gamma$. According to (EM) and $(\Gamma 3)$, $\neg A \in \Gamma \dot{\vee} {\sim}\neg A \in \Gamma$. Since $\neg A \notin \Gamma$, ${\sim}\neg A \in \Gamma$. By the rule $({\sim}\neg E)$ $\neg {\sim}A \in \Gamma$. According to (EM) and $(\Gamma 3)$, $A \in \Gamma \dot{\vee} {\sim}A \in \Gamma$. Since $A \notin \Gamma$, ${\sim}A \in \Gamma$. Hence, $e({\sim}A,\Gamma) = b = f_{\sim}(n) = f_{\sim}(e(A,\Gamma))$.

(C) $e(A,\Gamma) = b$. $A \in \Gamma$, $\neg A \in \Gamma$. Let ${\sim}A \in \Gamma$. Then by the rule (EFQ) $B \in \Gamma$, that is $\Gamma = Form^{\#}$, that contradicts to $(\Gamma 1)$. ${\sim}A \notin \Gamma$. Let $\neg {\sim}A \in \Gamma$. By the rule $(\neg {\sim}E)$ ${\sim}\neg A \in \Gamma$, by the rule (EFQ) $B \in \Gamma$. Contradiction. $\neg {\sim}A \notin \Gamma$. Hence, $e({\sim}A,\Gamma) = n = f_{\sim}(b) = f_{\sim}(e(A,\Gamma))$.

(D) $e(A,\Gamma) = 1$. $A \in \Gamma$, $\neg A \notin \Gamma$. Let ${\sim}A \in \Gamma$. Then by the rule (EFQ) $B \in \Gamma$, that is $\Gamma = Form^{\#}$, that contradicts to $(\Gamma 1)$. ${\sim}A \notin \Gamma$. According to (EM) and $(\Gamma 3)$, $\neg A \in \Gamma \dot{\vee} {\sim}\neg A \in \Gamma$. Since $\neg A \notin \Gamma$, ${\sim}\neg A \in \Gamma$. By the rule $({\sim}\neg E)$ $\neg {\sim}A \in \Gamma$. Hence, $e({\sim}A,\Gamma) = 0 = f_{\sim}(1) = f_{\sim}(e(A,\Gamma))$.

(3) (A) $e(A,\Gamma) = 0$, $e(B,\Gamma) = 0$. $A \notin \Gamma$, $\neg A \in \Gamma$, $B \notin \Gamma$, $\neg B \in \Gamma$. Suppose $A \vee B \in \Gamma$. According to $(\Gamma 3)$, $A \in \Gamma \dot{\vee} B \in \Gamma$. Contradiction. Then $A \vee B \notin \Gamma$. According to $(\wedge I)$ and $(\neg \vee I)$, $\neg(A \vee B) \in \Gamma$. Hence, $e(A \vee B,\Gamma) = 0 = f_{\vee}(0,0) = f_{\vee}(e(A,\Gamma), e(B,\Gamma))$.

(B) $e(A,\Gamma) = n$, $e(B,\Gamma) = b$. $A \notin \Gamma$, $\neg A \notin \Gamma$, $B \in \Gamma$, $\neg B \in \Gamma$. By the rule $(\vee I_2)$, $A \vee B \in \Gamma$. Suppose $\neg(A \vee B) \in \Gamma$, then by the rule $(\neg \vee E)$, $\neg A \wedge \neg B \in \Gamma$, but by the rule $(\wedge E_1)$, $\neg A \in \Gamma$. Contradiction. Hence, $\neg(A \vee B) \notin \Gamma$. Consequently, $e(A \vee B,\Gamma) = 1 = f_{\vee}(n,b) = f_{\vee}(e(A,\Gamma), e(B,\Gamma))$.

The other cases are proved similarly.

(4) (A) $e(A,\Gamma) = 0$, $e(B,\Gamma) = n$. $A \notin \Gamma$, $\neg A \in \Gamma$, $B \notin \Gamma$, $\neg B \notin \Gamma$. Suppose $A \wedge B \in \Gamma$. Then by the rules $(\wedge E_1)$ and $(\wedge E_2)$, $A \in \Gamma$ and $B \in \Gamma$. Contradiction. $A \wedge B \notin \Gamma$. By the rules $(\vee I_1)$ and $(\neg \wedge I)$, $\neg(A \wedge B) \in \Gamma$. Hence, $e(A \wedge B,\Gamma) = 0 = f_{\wedge}(0,n) = f_{\wedge}(e(A,\Gamma), e(B,\Gamma))$.

(B) $e(A,\Gamma) = 1$, $e(B,\Gamma) = 1$. $A \in \Gamma$, $\neg A \notin \Gamma$, $B \in \Gamma$, $\neg B \notin \Gamma$. By the rule $(\wedge I)$, $A \wedge B \in \Gamma$. Suppose $\neg(A \wedge B) \in \Gamma$. By the rule $(\neg \wedge E)$, $\neg A \vee \neg B \in \Gamma$, but then, according to $(\Gamma 3)$, $\neg A \in \Gamma \dot{\vee} \neg B \in \Gamma$. Contradiction. Hence, $\neg(A \wedge B) \notin \Gamma$. Consequently, $e(A \wedge B,\Gamma) = 1 = f_{\wedge}(1,1) = f_{\wedge}(e(A,\Gamma), e(B,\Gamma))$.

The other cases are proved similarly.

(5) (A) Let $e(A, \Gamma) = 0$. Then $A \notin \Gamma$, $\neg A \in \Gamma$.

(α) Suppose $e(\star A, \Gamma) = 0$. Then $f_\star(0) = 0$ and $\mathcal{R}_\star(0, 0)$ is a rule for $\star$ in **FDE#**. According to (EM) and ($\Gamma 3$), $A \in \Gamma \dot{\vee} \sim A \in \Gamma$. Since $A \notin \Gamma$, $\sim A \in \Gamma$. Then by the rules $\mathcal{R}_\star(0, 0)$, $(\wedge E_1)$ and $(\wedge E_2)$ $\sim\star A \in \Gamma$ and $\neg\star A \in \Gamma$. Let $\star A \in \Gamma$. Then by the rule (EFQ) $B \in \Gamma$, that contradicts to ($\Gamma 1$). $\star A \notin \Gamma$. Hence, $e(\star A, \Gamma) = 0 = f_\star(0) = f_\star(e(A, \Gamma))$.

(β) Suppose $e(\star A, \Gamma) = n$. Then $f_\star(0) = n$ and $\mathcal{R}_\star(0, n)$ is a rule for $\star$ in **FDE#**. Using (EM) and ($\Gamma 3$), obtain that $\sim A \in \Gamma$. By the rules $\mathcal{R}_\star(0, n)$, $(\wedge E_1)$ and $(\wedge E_2)$ $\sim\star A \in \Gamma$ and $\sim \neg \star A \in \Gamma$. Using (EFQ), obtain that $\star A \notin \Gamma$ and $\neg \star A \notin \Gamma$. Hence, $e(\star A, \Gamma) = n = f_\star(0) = f_\star(e(A, \Gamma))$.

(γ) Suppose $e(\star A, \Gamma) = b$. Then $f_\star(0) = b$ and $\mathcal{R}_\star(0, b)$ is a rule for $\star$ in **FDE#**. Clearly, that $\sim A \in \Gamma$. By the rules $\mathcal{R}_\star(0, b)$, $(\wedge E_1)$ and $(\wedge E_2)$ $\star A \in \Gamma$ and $\neg \star A \in \Gamma$. Hence, $e(\star A, \Gamma) = b = f_\star(0) = f_\star(e(A, \Gamma))$.

(δ) Suppose $e(\star A, \Gamma) = 1$. Then $f_\star(0) = 1$ and $\mathcal{R}_\star(0, 1)$ is a rule for $\star$ in **FDE#**. Clearly, that $\sim A \in \Gamma$. By the rules $\mathcal{R}_\star(0, 1)$, $(\wedge E_1)$ and $(\wedge E_2)$ $\star A \in \Gamma$ and $\sim\neg\star A \in \Gamma$. By the rule (EFQ) $B \in \Gamma$, $\neg\star A \notin \Gamma$. Hence, $e(\star A, \Gamma) = 1 = f_\star(0) = f_\star(e(A, \Gamma))$.

The other cases are proved similarly.

(6) Analogues to (5).

□

LEMMA 2. For every prime theory Γ and for every valuation v_Γ such that $\underset{p \in Prop}{\dot{\forall} p}\ (v_\Gamma(p) = e(p, \Gamma))$: $\underset{A \in Form^\#}{\dot{\forall} A}\ (v_\Gamma(A) = e(A, \Gamma))$.

PROOF. By structural induction on $\mathcal{L}^\#$-formula A using the lemma 1. □

LEMMA 3 (LINDENBAUM). For every set of $\mathcal{L}^\#$-formulas Γ, for every $\mathcal{L}^\#$-formula A: if $\Gamma \nvdash A$, then $\dot{\exists}\Gamma^*$: $\Gamma^* \subseteq Form^\#$ and (1) $\Gamma \subseteq \Gamma^*$, (2) $\Gamma^* \nvdash A$ and (3) Γ^* is a prime theory.

PROOF. Let $B_1, B_2, ...$ be an enumeration of all $\mathcal{L}^\#$-formulas. Now define a sequence of sets of $\mathcal{L}^\#$-formulas $\Gamma_1, \Gamma_2, ...$. Let $\Gamma_1 = \Gamma$ and Γ_n somehow defined. Then let $\Gamma_{n+1} = \Gamma_n \bigcup \{B_{n+1}\}$, if $\Gamma_n \bigcup \{B_{n+1}\} \nvdash A$; and $\Gamma_{n+1} = \Gamma_n$ otherwise. Let Γ^* is the union of all Γ_i.

(1) Follows from the definition of Γ^*.

(2) I will use the straightforward induction on i. Since $\Gamma_1 = \Gamma$, $\Gamma_1 \nvdash A$. By the inductive assumption, $\Gamma_i \nvdash A$. If $\Gamma_{i+1} = \Gamma_i$, then $\Gamma_{i+1} \nvdash A$. If $\Gamma_{i+1} \neq \Gamma_i$, then $\Gamma_{i+1} = \Gamma_i \bigcup \{B_{i+1}\}$. Suppose $\Gamma_i \bigcup \{B_{i+1}\} \vdash A$. But then (by the definition of the sequence of $\Gamma_1, \Gamma_2, ...$) $\Gamma_{i+1} = \Gamma_i$. Contradiction. Hence, $\Gamma_i \bigcup \{B_{i+1}\} \nvdash A$. Thus, if $\Gamma_{i+1} \neq \Gamma_i$, then $\Gamma_{i+1} \nvdash A$. Clearly, that if for all Γ_i true, that $\Gamma_i \nvdash A$, then $\Gamma^* \nvdash A$.

(3) Let us prove that: (A) $\Gamma^* \neq Form^\#$ (non-triviality); (B) $\Gamma^* \vdash B \Leftrightarrow B \in \Gamma^*$ (closure of $\vdash$); (C) $B \vee C \in \Gamma^* \Rightarrow (B \in \Gamma^* \dot{\vee} C \in \Gamma^*)$ (primeness).

(A) Since $\Gamma^* \nvdash A$, obviously, that $\Gamma^* \neq Form^\#$.

(B) ($\Rightarrow$). Suppose $\Gamma^* \vdash B$. Then $\dot{\exists} i$: $B = B_i$ and $\dot{\exists} \Gamma_i$: $\Gamma_i \vdash B_i$. Suppose $B_i \notin \Gamma_i$. Hence, $\Gamma_{i-1} \bigcup \{B_i\} \vdash A$. But then $\Gamma^* \vdash A$, because $\Gamma_{i-1} \subseteq \Gamma^*$ and $\Gamma^* \vdash B$. Nonetheless, it was proved in (2) that $\Gamma^* \nvdash A$. Then $B_i \in \Gamma_i$. Thus, $\Gamma^* \vdash B \Rightarrow B \in \Gamma^*$.

($\Leftarrow$). Suppose $B \in \Gamma^*$, $\Gamma^* \nvdash B$. Then $\dot{\exists} i$: $B = B_i$ and $\dot{\exists} \Gamma_{i-1}$: $\Gamma_{i-1} \bigcup \{B_i\} \vdash A$. Since $\Gamma_{i-1} \subseteq \Gamma^*$, $\Gamma^* \bigcup \{B_i\} \vdash A$. From here and the fact, that $\Gamma^* \nvdash A$, obtain, that $B_i \notin \Gamma^*$, that is $B \notin \Gamma^*$. Contradiction. Hence, $\Gamma^* \vdash A$. Thus, $B \in \Gamma^* \Rightarrow \Gamma^* \vdash A$.

(C) Suppose $B \vee C \in \Gamma^*$, but $B \notin \Gamma^*$, $C \notin \Gamma^*$. Since $B \vee C \in \Gamma^*$, $\Gamma^* \vdash B \vee C$ (see (B)). On the other hand, $\dot{\exists} i$: $B = B_i$ and $\dot{\exists} j$: $C = B_j$; $\Gamma_{i-1} \bigcup \{B_i\} \vdash A$ and $\Gamma_{j-1} \bigcup \{B_j\} \vdash A$. Moreover, $\Gamma_{i-1} \subseteq \Gamma^*$ and $\Gamma_{j-1} \subseteq \Gamma^*$. Then $\Gamma^* \bigcup \{B_i\} \vdash A$ and $\Gamma^* \bigcup \{B_j\} \vdash A$. From here and the fact, that $\Gamma^* \vdash B_i \vee B_j$, by the rule ($\vee E$) obtain, that $\Gamma^* \vdash A$, but according to (2), $\Gamma^* \nvdash A$. Hence, $B \vee C \in \Gamma^* \Rightarrow (B \in \Gamma^* \dot{\vee} C \in \Gamma^*)$.

□

THEOREM 2 (COMPLETENESS). *For every set of $\mathcal{L}^\#$-formulas Γ and for every $\mathcal{L}^\#$-formula A: $\Gamma \models A \Rightarrow \Gamma \vdash A$.*

PROOF. By contraposition. Let $\Gamma \nvdash A$. Then, by lemma 3, $\dot{\exists} \Gamma^* (\Gamma \subseteq \Gamma^*$, $\Gamma^* \nvdash A$ and Γ^* is a prime theory). According to lemma 2, there is a valuation v_{Γ^*} such, that $\underset{B \in \Gamma}{\dot{\forall} B}\, v_{\Gamma^*}(B) \in \{1, b\} \mathbin{\dot{\wedge}} v_{\Gamma^*}(A) \notin \{1, b\}$. But then $\Gamma \not\models A$. □

THEOREM 3 (ADEQUACY). *For every set of $\mathcal{L}^\#$-formulas Γ and for every $\mathcal{L}^\#$-formula A: $\Gamma \models A \Leftrightarrow \Gamma \vdash A$.*

PROOF. The theorem follows from the theorems 1 and 2. □

6. Natural deduction for implicative extensions of FDE

6.1. History and semantics

Using the technique of correspondence analysis, it is possible to axiomatize extensions of **FDE**, for example, implicative: $\mathbf{BN_4}$, **Par**, **FDEA**, **FDEB**, **FDEC**, and **FDED**. Connective $\mapsto$ is implication of the logic $\mathbf{BN_4}$, $\to_e$ is implication of the logic **Par**, $\to_a$ is implication of the logic **FDEA**, $\to_b$ is implication of the logic **FDEB**, $\to_c$ is implication of the logic **FDEC**, and $\to_d$ is implication of the logic **FDED**.

$\mapsto$	1	b	n	0
1	1	0	n	0
b	1	b	n	0
n	1	n	1	n
0	1	1	1	1

$\to_e$	1	b	n	0
1	1	b	n	0
b	1	b	n	0
n	1	1	1	1
0	1	1	1	1

$\to_a$	1	b	n	0
1	1	b	n	0
b	1	1	n	n
n	1	1	1	1
0	1	1	1	1

$\to_b$	1	b	n	0
1	1	b	n	0
b	1	1	n	n
n	1	b	1	b
0	1	1	1	1

$\to_c$	1	b	n	0
1	1	b	n	0
b	1	b	n	0
n	1	b	1	b
0	1	1	1	1

$\to_d$	1	b	n	0
1	1	b	n	0
b	1	b	n	0
n	1	b	1	b
0	1	b	1	b

A	$\neg_e$	$\neg_a$	$\neg_c$	$\neg_d$
1	0	0	0	0
b	0	n	0	0
1	1	b	b	b
0	1	1	1	b

The logic $\mathbf{BN_4}$ first appeared in R.T. Brady's paper [6], where several semantics for it and a Hilbert-style calculus are introduced. There is another reference of this logic (independent of [6]) in J.K. Slaney's paper [19].

The logic **Par** was first formulated by V.M. Popov [14] in the form of sequent and Hilbert-style calculuses. A similar Hilbert-style system independently appeared in A. Avron's paper [2] under the name **HBe**. Avron also introduced four-valued semantics for $\to_e$ [2]. Moreover, functional equivalence of $\to_e$ and $\mapsto$ was proven in [2]. Furthermore, the truth table for $\to_e$ is mentioned in A.P. Pynko's paper [17] in relation to [14], but independent of [2]. In addition, it is easy to see that $A \to_e B \equiv_{def} \neg_e A \vee B$[4].

[4] M. De and H. Omori [7] investigated four-valued classical negations $\neg_e$, $\neg_a$, $\neg_c$ and $\neg_d$ (in the notation of [7] $\neg_e$, $\neg_1$, $\neg_2$ and $\neg_5$) in line with the study of the relationship

Using negations $\neg_a$, $\neg_c$ and $\neg_d$, it's possible to define implications of logics **FDEA**, **FDEC**[5] and **FDED**: $A \to_a B \equiv_{def} \neg_a A \vee B$; $A \to_c B \equiv_{def} \neg_c A \vee B$; $A \to_d B \equiv_{def} \neg_d A \vee B$.

A semantics of the logic **FDEB** is first explored in D.V. Zaitsev's doctoral dissertation [21]. Notice that $A \to_b B \equiv_{def} \sim A \vee B$.

It is noteworthy that in the paper [7] by M. De and H. Omori a logic **BD+** with connectives $\neg$, $\sim$, $\vee$, $\wedge$ and $\to_b$ is analyzed.

It is easy to see that for all i ($i \in \{a, b, c, d, e\}$) $A, A \to_i B \models B$, $\models A \to_i (B \to_i A)$, $\models (A \to_i (B \to_i C)) \to_i ((A \to_i B) \to_i (A \to_i C))$, and $\models ((A \to_i B) \to_i A) \to_i A$. Thus, implications $\to_a$, $\to_b$, $\to_c$, $\to_d$ and $\to_e$ are classical.

6.2. Rules of inference

Using the proposition 2 and the theorem 3, it is not difficult to find necessary rules of inference for $\to_i$ ($i \in \{a, b, c, d, e\}$). Nonetheless, it makes sense to reduce the number of the rules. As a result, natural deduction systems will become more convenient for work in them. It is possible to prove that the rules for $\mapsto$ can be reformulated as follows[6]:

$$(\mapsto I_1)\ \frac{\neg A,\ B}{A \mapsto B} \quad (\mapsto I_2)\ \frac{\neg A}{A \vee (A \mapsto B)} \quad (\mapsto I_3)\ \frac{}{A \vee \neg B \vee (A \mapsto B)}$$

$$(\mapsto I_4)\ \frac{B}{\neg B \vee (A \mapsto B)} \quad (MP)\ \frac{A,\ A \mapsto B}{B} \quad (MT)\ \frac{A \mapsto B,\ \neg B}{\neg A}$$

$$(\neg \mapsto I)\ \frac{A,\ \neg B}{\neg (A \mapsto B)} \quad (\neg \mapsto E)\ \frac{\neg (A \mapsto B)}{A \wedge \neg B}$$

Logics **Par**, **FDEA**, **FDEB**, **FDEC** and **FDED** contain the following rules in common:

$$(\to I_1)\ \frac{B}{A \to B} \quad (\to I_2)\ \frac{}{A \vee (A \to B)} \quad (MP)\ \frac{A,\ A \to B}{B}$$

The axiomatization of **Par** contains also the following rules:

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of classical negation and properties of paraconsistency and paracompleteness of logical systems.

[5]Logics **FDEA** and **FDEC** have two relatives: **BD1** with the connectives $\neg$, $\neg_a$, $\to_a$, $\vee$ and $\wedge$; and **BD2** with the connectives $\neg$, $\neg_c$, $\to_c$, $\vee$ and $\wedge$ [7].

[6]There are two ways of proving this statement: (1) by proving the deductive equivalence of modified rules and rules based on the proposition 2; or (2) by completeness proof for implications just as it was done for the other connectives in the section 5.

$$(\neg \to_e I)\ \frac{A,\ \neg B}{\neg(A \to_e B)} \qquad (\neg \to_e E)\ \frac{\neg(A \to_e B)}{A \wedge \neg B}$$

The axiomatization of **FDEA** contains also the following rules:

$$(\neg \to_a I)\ \frac{A,\ \neg B}{\neg A \vee \neg(A \to_a B)} \qquad (\neg \to_a E_1)\ \frac{\neg(A \to_a B)}{A \wedge \neg B}$$

$$(\neg \to_a E_2)\ \frac{\neg(A \to_a B),\ \neg A}{C}$$

The axiomatization of **FDEB** contains also the following rules:

$$(\neg \to_b I)\ \frac{\neg B}{\neg A \vee \neg(A \to_b B)} \qquad (\neg \to_b E_1)\ \frac{\neg(A \to_b B)}{\neg B}$$

$$(\neg \to_b E_2)\ \frac{\neg(A \to_b B),\ \neg A}{C}$$

The axiomatization of **FDEC** contains also the following rules:

$$(\neg \to_c I_1)\ \frac{A,\ \neg B}{\neg(A \to_c B)} \qquad (\neg \to_c I_2)\ \frac{\neg B}{\neg A \vee \neg(A \to_c B)}$$

$$(\neg \to_c E_1)\ \frac{\neg(A \to_c B)}{\neg B} \qquad (\neg \to_c E_2)\ \frac{\neg(A \to_c B),\ \neg A}{A}$$

The axiomatization of **FDED** contains also the following rules:

$$(\neg \to_d I_1)\ \frac{A,\ \neg B}{\neg(A \to_d B)} \qquad (\neg \to_d I_2)\ \frac{\neg B}{\neg A \vee \neg(A \to_d B)}$$

$$(\neg \to_d I_3)\ \frac{\neg A,\ \neg B}{A \vee \neg(A \to_d B)} \qquad (\neg \to_d E)\ \frac{\neg(A \to_d B)}{\neg B}$$

7. Conclusion

In summary, the result obtained in this paper allows to get immediately adequate natural deduction systems for all possible truth-table expansions of **FDE**+. Consequently, a problem for future research arises: to formulate propositions 1 and 2 without the use of Boolean negation, in other words, to apply the technique of correspondence analysis to **FDE** directly, without recourse to **FDE**+. In future prospect one more direction of research opens: to apply the technique of correspondence analysis to other four-valued logics or even to arbitrary k-valued logics with l designated values, where $k \geqslant 3$ and $l \in \{1, ..., k-1\}$.

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